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Wealth Distribution: To be or not to be a Gamma?

ABSTRACT : We review some aspects, especially those we can tackle analytically, of a minimal model of closed economy analogous to the kinetic theory model of ideal gases where the agents exchange wealth amongst themselves such that the total wealth is conserved, and each individual agent saves a fraction (0 ≤ λ ≤ 1) of wealth before transaction. We are interested in the special case where the fraction λ is constant for all the agents (global saving propensity) in the system. We show by moment calculations that the resulting wealth distribution cannot be the Gamma distribution that was conjectured in Phys. Rev. E 70, 016104 (2004). We also derive an upper-bound form for the distribution at low wealth, which is a new result.

The distribution of wealth or income in society has been of great interest for many years. As first noticed by Pareto in the 1890's[1], the wealth distribution seems to follow a "natural law" where the tail of the distribution is described by a power-law $f(x) \sim x^{-(1+\alpha)}$. Away from the tail, the distribution is better described by a Gamma or Log-normal distribution known as Gibrat's law[2]. Considerable investigation with real data during the last ten years revealed that the power-law tail exhibits a remarkable spatial and temporal stability and the Pareto index α is found to have a value between 1 and 2 (see Refs.[3,4]). Even after 110 years the origin of the power-law tail remained unexplained but recent interest of physicists and mathematicians in econophysics has led to a new insight into this problem (see Refs.[5-7]).

Our general aim is to study a many-agent statistical model of closed economy (analogous to the kinetic theory model of ideal gases)[8–13], where N agents exchange a quantity x, that may be defined as wealth. The states of agents are characterized by the wealth $\{x_i\}$, i = 1, 2, … , N, and the total wealth $W = \Sigma_i x_i$ is conserved. The evolution of the system is then carried out through pair-wise interactions characterized by a saving parameter λ, with $0 \le \lambda \le 1$. The equilibrium distribution of wealth $f(x)$ is defined as follows : $f(x)dx$ is the probability that in the steady state of the system, a randomly chosen agent will be found to have wealth between x and $x + dx$. In these models, if λ is equal for all the units, $f(x)$ is fitted quite well by a Gamma distribution[14-16]

$$f(x) = \frac{1}{\Gamma(n)} \left(\frac{n}{\langle x \rangle} \right)^n x^{n-1} \exp\left(-\frac{nx}{\langle x \rangle} \right), \tag{1}$$

where

$$n = \frac{D(\lambda)}{2} = 1 + \frac{3\lambda}{1-\lambda}. \tag{2}$$

This equilibrium distribution (1) had been suggested by an analogy with the kinetic theory of gases in $D(\lambda)$ dimensions[14-16].

In this paper we show by the method of moment calculations that the resulting wealth distribution cannot be the Gamma distribution that was conjectured in Ref.[15,16]. We also derive the functional form of an upper bound on $f(x)$ at very small x.

Many-agent Model of a Closed Economy

In the kinetic wealth exchange models, the system evolves according to a prescription, which defines the trading rule between agents. At every time step two agents i and j are extracted randomly and an amount of wealth Δx is exchanged between them,

$$x_i' = x_i - \Delta x,$$

$$x_j' = x_j - \Delta x. \tag{3}$$

It can be noticed that in this way, the quantity x is conserved during the single transactions: $x_i' + x_j' = x_i + x_j$ (see Fig. 1), where x_i' and x_j' are the agent wealths after the transaction has taken place. Several simple models dealing with different transaction rules have been studied (see the reviews[5,6,17] and references therein). Here we will present a few examples.

Basic model without saving (Boltzmann distribution) : In the first version of the model, the wealth difference Δx is assumed to have a constant value[8],

$$\Delta x = \Delta x_0 . \tag{4}$$

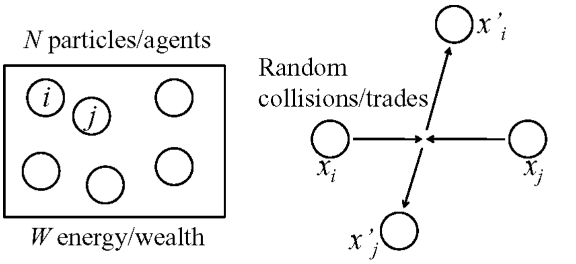

Fig. 1. Analogy of the minimal economic model with an isolated ideal gas, where the particles are randomly undergoing elastic collisions, and exchanging kinetic energy. In the closed economy, the economic agents randomly trade with each other according to some rule, and exchange wealth.

This rule, together with the constraint that transactions can take place only if $x_i' > 0$ and $x_j' > 0$, provides a Boltzmann distribution, see the curve for $\lambda = 0$ in Fig. 2. Alternatively, Δx can be a random fraction of the wealth of one of the two agents,

$$\Delta x = \varepsilon x_i \quad or \quad \Delta x = \varepsilon x_j , \tag{5}$$

where ε is a random number uniformly distributed between 0 and 1. A trading rule based on the random redistribution of the sum of the wealths of the two agents had been introduced by Dragulescu and Yakovenko[11],

$$x_i' = \varepsilon\left(x_i + x_j\right),$$

$$x_j' = \left(1-\varepsilon\right)\left(x_i + x_j\right). \tag{6}$$

Equations (6) are easily shown to correspond to the trading rule (3), with

$$\Delta x = \left(1-\varepsilon\right)x_i - \varepsilon x_j . \tag{7}$$

All the versions of the basic model lead to an equilibrium Boltzmann distribution, given by

$$f(x) = \frac{1}{\langle x \rangle}\exp\left(-\frac{x}{\langle x \rangle}\right), \tag{8}$$

where the effective temperature of the system is just the average wealth $\langle x \rangle$[8,11]. The result (8) is found to be robust; it is largely independent of various factors. Namely, it is obtained for the various forms of Δx mentioned above, for a pair-wise as well as multi-agent interactions, for arbitrary initial conditions[12], and finally, for random or consecutive extraction of the interacting agents. For the trading rule (6) one can show the convergence towards the Boltzmann distribution through different methods: Boltzmann equation, entropy maximization, distributional equation, *etc*.

Model with global saving propensity λ : A step toward generalizing the basic model (3) and making it more realistic, is the introduction of a saving criterion regulating the trading dynamics. This can be practically achieved by defining a saving propensity $0 \le \lambda \le 1$, which represents the fraction of wealth which is saved – and not reshuffled – during a transaction. The dynamics of the model is as follows[12, 18] :

$$x_i' = \lambda x_i + \varepsilon(1-\lambda)\left(x_i + x_j\right),$$

$$x_j' = \lambda x_j + (1-\varepsilon)\,(1-\lambda)\,(x_i + x_j). \tag{9}$$

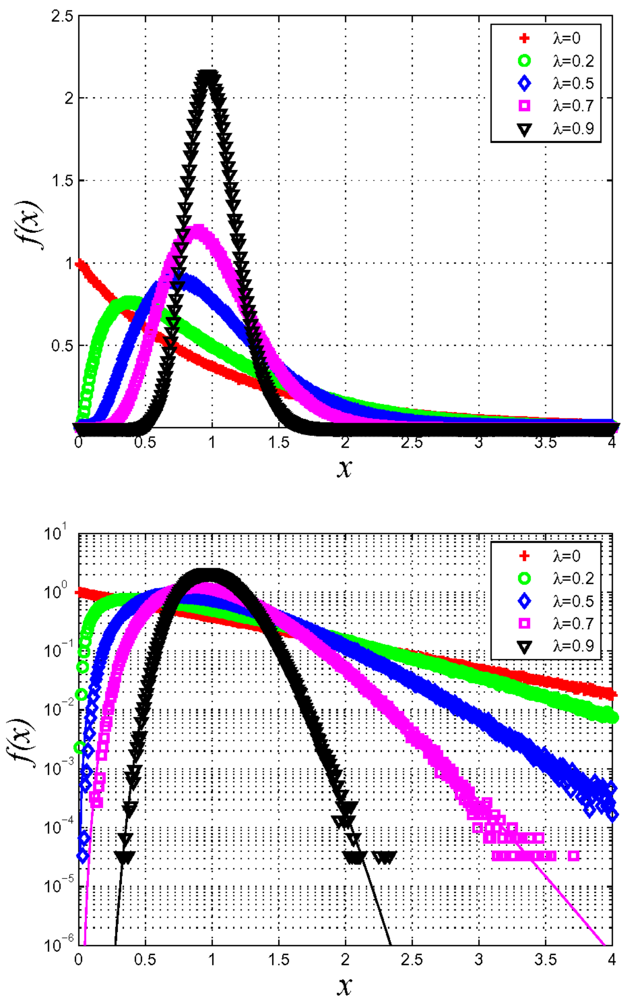

Fig. 2. Probability density for wealth x. The curve for $\lambda = 0$ is the Boltzmann function $f(x) = \langle x \rangle^{-1}\exp\left(-x/\langle x \rangle\right)$ for the basic model. The other curves correspond to a global saving propensity $\lambda > 0$.

corresponding to a Δx in Eq. (3) given by

$$\Delta x = (1-\lambda)\left[(1-\varepsilon)x_i - \varepsilon x_j\right]. \tag{10}$$

This model leads to a qualitatively different equilibrium distribution. In particular, it has a mode $x_m > 0$ and $f(0) = 0$; see Fig. 2. Later we will derive a form for an upper bound on $f(x)$ at low range. The functional form of such a distribution was conjectured to be a Gamma-distribution, as given by Eq. (1) on the basis of an analogy with the kinetic theory of gases. Indeed, it is easy to show, starting from the Maxwell-Boltzmann distribution for the particle velocity in a D dimensional gas, that the equilibrium kinetic energy distribution coincides with the Gamma-distribution (1) with $n = \frac{D}{2}$. This conjecture is remarkably consistent with the fitting provided to numerical data[14-16]. In the following section we will show by two different approaches that the conjecture (1) cannot be the actual equilibrium distribution.

Analytical Results for Model with Saving Propensity

Fixed-point distribution : In order to obtain analytical results we are interested in the thermodynamic limit $(N \to \infty)$. When the number of agents is very large, a particular agent will interact with another particular agent very rarely (because agents are chosen randomly with same probability). Thus, the agents can be considered independent. Then at equilibrium, while the agents wealth follow the dynamics of Eq.(9), the global distribution does not change, so we can write:

$$X \stackrel{d}{=} \lambda X_1 + \varepsilon(1-\lambda)\left(X_1 + X_2\right), \tag{11}$$

where $\stackrel{d}{=}$ means identity in distribution and one assumes that the random variables X_1, X_2 and X have the same probability law, while the variables X_1, X_2 and X are stochastically independent. It seems dificult to find the distribution of X, however, one can compute the moments of f. Indeed with (11), one can write immediately

$$\forall m \in \mathbb{N}, \left\langle X^m \right\rangle = \left\langle \left(\lambda X_1 + \varepsilon(1-\lambda)(X_1 + X_2)\right)^m \right\rangle, \tag{12}$$

and by developing (12) one can find the recursive relation

$$\left\langle X^m \right\rangle = \sum_{k=0}^{m} \binom{m}{k} \frac{\lambda^{m-k}(1-\lambda)^k}{k+1} \sum_{p=0}^{k} \binom{k}{p} \left\langle X^{m-p} \right\rangle \left\langle X^p \right\rangle. \tag{13}$$

Using (13) with initial conditions $\left\langle X^0 \right\rangle = 1$ (normalization) and $\left\langle X^1 \right\rangle = 1$ (without loss of generality), we obtain

$$\left\langle X^2 \right\rangle = \frac{\lambda+2}{1+2\lambda}, \tag{14}$$

$$\left\langle X^3 \right\rangle = \frac{3(\lambda+2)}{(1+2\lambda)^2}, \tag{15}$$

$$\left\langle X^4 \right\rangle = \frac{72+12\lambda-2\lambda^2+9\lambda^3-\lambda^5}{(1+2\lambda)^2(3+6\lambda-\lambda^2+2\lambda^3)}. \tag{16}$$

Now let us compare these moments with conjecture (1)'s moments. Setting $\langle x \rangle = 1$ in Eq. (1) it is easy to show

$$\left\langle x^k \right\rangle = \frac{(n+k-1)\,(n+k-2)\,...\,(n+1)}{n^{k-1}}. \tag{17}$$

Writing (17) for k = 2, 3, 4 and choosing n as in (2) we find

$$\left\langle x^2 \right\rangle = \frac{n+1}{n} = \frac{\lambda+2}{1+2\lambda}, \tag{18}$$

$$\left\langle x^3 \right\rangle = \frac{(n+2)(n+1)}{n^2} = \frac{3(\lambda+2)}{(1+2\lambda)^2}, \tag{19}$$

$$\left\langle x^4 \right\rangle = \frac{(n+3)(n+2)(n+1)}{n^3} = \frac{3(\lambda+2)\,(4-\lambda)}{(1+2\lambda)^3}. \tag{20}$$

The fourths moments (eqs.(16) and (20)) are different so the conjecture that the Gamma distribution is an equilibrium solution of this model is wrong. Nevertheless the first three moments coincide exactly which shows that the Gamma-distribution is strangely a very good approximation. Moreover the deviation in the fourth moment is very small (see Fig. 3, which shows that the two curves can hardly be distinguished by the naked eye). Finding a function that would coincide to higher moments is still an open challenge. These results are consistent with the ones found by Repetowicz *et al.*[19] which will be presented in the following sub-section.

Laplace transform analysis : In this section we will confirm the previous result with a different approach based on the Boltzmann equation and along the lines of Bassetti *et al.*[20]. Given a fixed number of N agents in a system, which are allowed to trade, the interaction rules describe a stochastic process of the vector variable $\left(x_1(\tau) \,......, x_N(\tau)\right)$ in discrete time τ. Processes of this type are thoroughly studied e.g. in the context of kinetic theory of ideal gases.

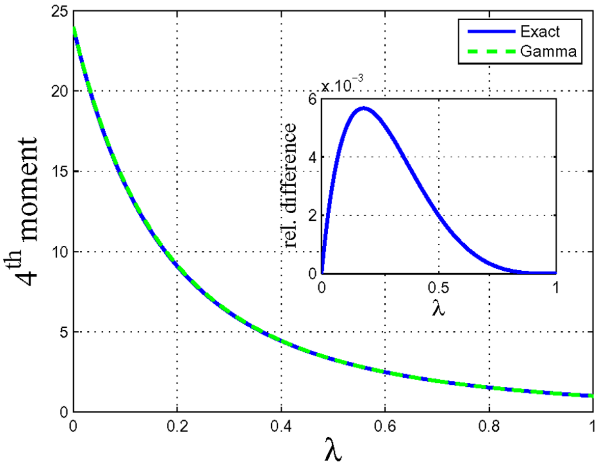

Fig. 3. Exact fourth moment eq.(16) and Gamma distribution fourth moment eq.(20) against λ. The inset shows the relative dierence between exact fourth moment eq.(16) and Gamma distribution fourth moment eq.(20) against λ.

Indeed, if the variables x_i are interpreted as energies corresponding to the *i*-th particle, one can map the process to the mean-field limit of the Maxwell model of particles undergoing random elastic collisions. The full information about the process in time τ is contained in the *N*-particle joint probability distribution $P_N\left(\tau, x_1, x_2, \dots, x_N\right)$. However, one can write a kinetic equation for one-marginal distribution function

$$P_1(\tau, x) = \int P_N\left(\tau, x, x_2, \dots, x_N\right) dx_2 \dots dx_N ,$$

involving only one- and two-particle distribution functions

$$P_1(\tau+1, x) - P_1(\tau, x) = \left\langle \frac{1}{N}\left[\int P_2\left(\tau, x_i, x_j\right)\right.\right.$$
$$\left(\delta\left(x - \lambda x - (1-\lambda)\,\varepsilon(x_i + x_j)\right)\right)$$
$$\left.\left. +\delta\left(x - \lambda x - (1-\lambda)(1-\varepsilon)\,(x_i + x_j)\right)\right) dx_i dx_j - 2P_1(\tau, x)\right]\right\rangle,$$

which may be continued to give eventually an infinite hierarchy of equations of BBGKY (Born, Bogoliubov, Green, Kirkwood, Yvon) type[21]. In the thermodynamic limit, the agents becomes independent, as explained earlier. Thus, we can write

$$P_2\left(\tau, x_i, x_j\right) = P_1(\tau, x_i)\, P_1(\tau, x_j) ,$$

which implies a closure of the hierarchy at the lowest level. Therefore, the one-particle distribution function bears all information. Rescaling the time as $t = \frac{2\tau}{N}$ in the thermodynamic limit $N \to \infty$, one obtains for the one-particle distribution function *f*(*t*, *x*) the Boltzmann-type kinetic equation

$$\frac{\partial f(t,x)}{\partial t} = \frac{1}{2}\left\langle \int f(t,x_i)\, f(t,x_j)\left(\delta\left(x - \lambda x - (1-\lambda)\varepsilon(x_i + x_j)\right)\right.\right.$$
$$\left.\left. +\delta\left(x - \lambda x - (1-\lambda)(1-\varepsilon)(x_i + x_j)\right)\right) dx_i dx_j \right\rangle - f(t,x). \quad (21)$$

This equation can be written (see Matthes *et al.*[20]) as

$$\frac{\partial f(t,x)}{\partial t} = Q(f, f),$$

where Q is a *collision operator.* A collision operator is bilinear and satisfies, for all smooth functions $\phi(x)$

$$\int_0^\infty Q(f, f)\, \phi(x) dx$$
$$= \frac{1}{2}\left\langle \int_0^\infty \int_0^\infty \left(\phi\left(x_i'\right) + \phi\left(x_j'\right) - \phi\left(x_i\right) - \phi\left(x_j\right)\right)\right.$$
$$\left. f\left(x_i\right) f\left(x_j\right) dx_i dx_j \right\rangle, \quad (22)$$

where x_i' and x_j' are the post-trade wealth. With this property the equation (21) can be written in the weak form, for all smooth functions $\phi(x)$

$$\frac{d}{dt}\int_0^\infty f(t, x)\, \phi(x) dx$$
$$= \frac{1}{2}\left\langle \int_0^\infty \int_0^\infty \left(\phi\left(x_i'\right) + \phi\left(x_j'\right) - \phi\left(x_i\right) - \phi\left(x_j\right)\right)\right.$$
$$\left. f\left(x_i\right) f\left(x_j\right) dx_i dx_j \right\rangle. \quad (23)$$

It is very useful because the choice $\phi(x) = e^{-sx}$ gives (after some calculations) the Boltzmann equation for the Laplace transform $\hat{f}$ of f

$$\frac{\partial \hat{f}(t,x)}{\partial t} + \hat{f}(t,s) = \frac{1}{2}\left\langle \hat{f}\left(t, (\lambda + (1-\lambda)\varepsilon)s\right) \hat{f}\left(t, (1-\lambda)\varepsilon s\right)\right.$$
$$\left. +\hat{f}\left(t, (1-\lambda)(1-\varepsilon)s\right) \hat{f}\left(t, 1-(1-\lambda)\varepsilon s\right)\right\rangle. \quad (24)$$

For the steady state, and if ε is drawn randomly from a uniform distribution, the previous equation reduces to

$$s\hat{f}(s) = \frac{1}{1-\lambda}\int_0^{(1-\lambda)s} \hat{f}(\lambda s + y)\, \hat{f}(y) dy , \quad (25)$$

which coincides with results of Ref.[19]. The Taylor expansion of $\hat{f}(s)$ can be derived by substituting the expansion $\hat{f}(s)=\sum_{p=0}^{\infty}(-1)^p m_p s^p$ in (25). Since $\hat{f}(-s)$ is the moment-generating function, we have $\langle x^k \rangle = m_k \cdot k!$. With this method Repetowicz *et al.*[19] obtained the recursive formula

$$m_p = \sum_{q=0}^{p} m_q m_{p-q} \tilde{C}_q^{(p)}(\lambda) \tag{26}$$

with

$$\tilde{C}_q^{(p)}(\lambda) = \frac{\int_0^{(1-\lambda)} (\lambda+\eta)^q \, \eta^{p-q} d\eta}{1-\lambda},$$

$$\tilde{C}_{q+1}^{(p)} = \frac{(1-\lambda)^{p-q-1} - (q+1)\tilde{C}_q^{(p)}}{p-q},$$

$$\tilde{C}_0^{(p)} = \frac{(1-\lambda)^p}{p+1}. \tag{27}$$

Now with this formula one can obtain the first four moments and they match the ones found in the previous section Eqs. (14-16), which confirms that the Gamma-distribution is not the stationary distribution.

Upper Bound Form at Low Wealth Range

From equation (11)

$$X \overset{d}{=} \lambda X_i + \varepsilon(1-\lambda)\,(X_i + X_j),$$

we have for all $x \geq 0$

$$\mathcal{P}[X \leq x] = \mathcal{P}\left[\lambda X_i + \varepsilon(1-\lambda)\,(X_i+X_j) \leq x\right], \tag{28}$$

where $\mathcal{P}[.]$ means the probability of the event inside the brackets. Again, we consider that the agents are independent, which is true when $N \to \infty$. Then

$$\int_0^x dx\, f(x) = \int_0^\infty dx_i f(x_i) \int_0^\infty dx_j f(x_j)$$

$$\times \int_0^1 d\varepsilon \Theta\left[x - \lambda x_i + \varepsilon(1-\lambda)\,(x_i + x_j)\right], \tag{29}$$

where Θ is the Heaviside step function. Taking the derivative with respect to x in both sides, we have

$$f(x) = \int_0^\infty dx_i f(x_i) \int_0^\infty dx_j f(x_j)$$

$$\times \int_0^1 d\varepsilon \delta\left[x - \lambda x_i + \varepsilon(1-\lambda)\,(x_i + x_j)\right]. \tag{30}$$

This equation is an integral equation for *f*(*x*). As mentioned earlier, we are not able to solve it in closed form. However, one can simplify the equation, by doing the integral over ε. Then the δ-function will contribute only if we have the following constraints

$$0 \leq x_i \leq x/\lambda, \tag{31}$$

$$\frac{x - x_i}{1-\lambda} \leq x_j\,, \tag{32}$$

$$0 \leq x_j \tag{33}$$

The range defined by these constraints is shown in figure 4. In this range, the derivative of the argument of the delta function with respect to ε is $(x_i + x_j)(1-\lambda)$. Hence, we get

$$f(x) = \frac{1}{1-\lambda}\int_0^{x/\lambda} dx_i f(x_i) \int_{\max\left(\frac{x-x_i}{1-x},0\right)}^{\infty} dx_j f(x_j) \frac{1}{x_i + x_j}. \tag{34}$$

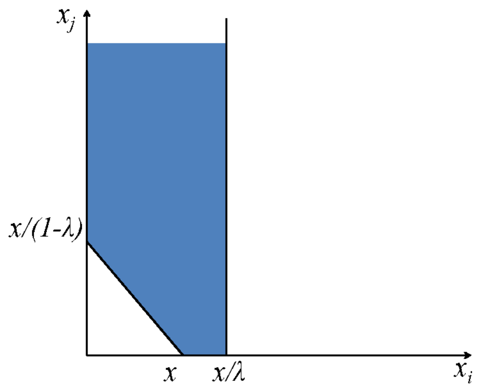

Fig. 4. Region of integration.

This immediately gives

$$f(x) \leq C \int_0^{x/\lambda} f(x_i)\, dx_i\,, \tag{35}$$

where

$$C = \frac{1}{1-\lambda}\int_0^\infty dx_j f(x_j) \frac{1}{x_j}. \tag{36}$$

We assume that *f* decays fast enough near 0 so that the integral in (36) is well defined. Now (35) may be rewritten by rescaling the variable, as

$$f(\lambda x) \leq C \int_0^x dx_i f(x_i). \tag{37}$$

We now use the observation that for $\lambda > 0$ the numerically determined $f(x)$ is a continuous function with a single maximum, say at x_0 (see Fig. 2). Then for all $x \leq x_0$, the integrand (37) takes its maximum value at the right extreme point, i.e. when $x_i = x$. This then gives us

$$f\left(\lambda x\right) \leq Cxf(x), \quad \text{for} \quad x \leq x_0 . \tag{38}$$

Iterating this equation, we get

$$f\left(\lambda^r x\right) \leq C^r \lambda^{r(r-1)/2} x^r f(x) . \tag{39}$$

We can set $x = x_0$ in the above equation, giving

$$f\left(\lambda^r x_0\right) \leq C^r \lambda^{r(r-1)/2} x_0^r f(x_0) . \tag{40}$$

Then taking $r \approx -\log x$ and rescaling the variables, we get

$$f(x) = \mathrm{O}\left(x^{\alpha} \exp\left[-\beta\,(\log x)^2\right]\right), \tag{41}$$

as $x \to 0$, where α and $\beta (> 0)$ are two constants dependent on λ. The Gamma-distribution decays slower than the rhs in (41) when $x \to 0$. The expression (41) gives an upper bound form at low wealth range and confirms again that the distribution of the global saving propensity model is not a Gamma-distribution.

Discussion and Summary

As a further generalization, the agents could be assigned different saving propensities λ_i [13, 19, 22–25]. In particular, uniformly distributed λ_i in the interval [0, 1) have been studied numerically in Refs.[13, 22]. This model is described by the trading rule

$$x_i' = \lambda_i x_i + \varepsilon\left[\left(1-\lambda_i\right)x_i + \left(1-\lambda_j\right)x_j\right],$$

$$x_j' = \lambda_j x_j + (1-\varepsilon)\left[\left(1-\lambda_i\right)x_i + \left(1-\lambda_j\right)x_j\right], \tag{42}$$

or, equivalently, by a Δx (as defined in Eq. (3)) given by

$$\Delta x = (1-\varepsilon)\left(1-\lambda_i\right)x_i - \varepsilon\left(1-\lambda_j\right)x_j . \tag{43}$$

One of the main features of this model, which is supported by theoretical considerations[19, 23, 26], is that the wealth distribution exhibits a robust power-law at large values of x,

$$f(x) \propto x^{-\alpha-1}, \tag{44}$$

with a Pareto exponent $\alpha = 1$ largely independent of the details of the λ-distribution. It may be noted that the exponent value unity is strictly for the tail end of the distribution and not for small values of the income or wealth (where the distribution remains exponential). Also, for finite number N of agents, there is always an exponential (in N) cut-off at the tail end of the distribution. We do not discuss in details such generalizations here, and for deeper interests a reader may refer to the review[27] or references therein.

In summary, we have used different approaches to show that the correct form of the wealth distribution cannot be the Gamma distribution. We have computed all the exact moments of this distribution. We have also derived the analytical form of an upper bound at low wealth range (see Eq. (41)). Nevertheless, the closed form of the solution to Eq. (9) still remains an open question.

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